Journal of Computational Physics 228 (2009) 3209–3231

Contents lists available at ScienceDirect

Journal of Computational Physics

journal homepage: www.elsevier.com/locate/jcp

# On the inverse scattering problem in the acoustic environment

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ARTICLE INFO

*Article history:*
Received 23 April 2008
Accepted 15 December 2008
Available online 14 January 2009

*Keywords:*
Inverse scattering
Helmholtz
Impedance
Trace formulae
Acoustic

ABSTRACT

In this paper, we construct numerical algorithms for the solution of inverse scattering problems in layered acoustic media. Our inverse scattering schemes are based on a collection of so-called trace formulae. The speed $c$ of propagation of sound, the density $\rho$, and the attenuation $\gamma$ are the three parameters reconstructed by the algorithm, given that all of them are laterally invariant. For a medium whose parameters $c, \rho$, and $\gamma$ have $m \geqslant 1$ continuous derivatives, and data measured for frequencies between 0 and $a > 0$, the error of our scheme decays as $1/a^{m-1}$ as $a \to \infty$. In this respect, the algorithm is similar to the Fourier Transform. Our results are illustrated with several numerical examples.



## 1. Introduction

Inverse scattering has been an active field of research in science, mathematics, and engineering over the past several decades (see e.g. [2,3,6,8,10–12,14]). It has applications in a wide range of fields, such as radar, medical imaging, oil exploration, microscopy, etc. In this paper, we construct numerical algorithms for the solution of inverse scattering problems in the acoustic environment in three dimensions. Our inverse scattering scheme assumes that the speed $c(x,y,z)$ of propagation of sound, the density $\rho(x,y,z)$ and the attenuation $\gamma(x,y,z)$ are independent of the variables $x,y$, so that $c(x,y,z) = c(z), \rho(x,y,z) = \rho(z), \gamma(x,y,z) = \gamma(z)$; an acoustic medium possessing these properties will be referred to as a layered medium, or layered environment.

The inverse scattering schemes we construct are based on a collection of so-called trace formulae, and can be viewed as extension of the work started in [3], where the observation is made that (at least in layered media) it is possible to construct inverse scattering algorithms that, given a smoothly varying medium, require few measurements to reconstruct it. More specifically, given a medium whose parameters $c, \rho$, and $\gamma$ have $m \geqslant 1$ continuous derivatives, and data measured for all frequencies $\omega$ on the interval $[-a,a]$, the error of the reconstruction decays as $1/a^{m-1}$ as $a \to \infty$. In this respect, the algorithm of [3] is similar to the Fourier Transform, and a strong argument is made that this is a very desirable property. While the algorithm of [3] assumes that the parameters $\rho$ and $\gamma$ are constant and the parameter $c$ depends on $z$, the schemes of this paper reconstruct $c, \rho$, and $\gamma$, provided that they only depend on the coordinate $z$.

The paper is organized as follows: Section 2 introduces the mathematical formulation of the problem. In Section 3, we summarize several well-known mathematical facts to be used in the paper. In Section 4, we introduce analytical tools to

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1 Supported in part by ONR Grant N00014-07-1-0674.
2 Supported in part by the Schlumberger Technology Corporation, ONR Grant N00014-07-1-0711, and AFOSR Grant FA9550-06-1-0239..

0021-9991/$ - see front matter 
doi:10.1016/j.jcp.2008.12.034

be used in the construction of the algorithm. Section 5 states the algorithm in detail, and a complexity analysis is included. In Section 6, several numerical examples are used to illustrate the performance of the algorithm.

## 2. Statement of the problem

The inverse scattering problem is the problem of reconstructing the various parameters of scattering objects, such as the density, the speed of sound, and the attenuation, with the knowledge of the incident and the scattered field. Below is the formal mathematical formulation of the three-dimensional inverse scattering problem in a layered acoustic medium.

### 2.1. The Helmholtz equation

The inverse scattering problem we investigate arises from the time domain wave equation

$$\frac{\partial^2}{\partial^2 t}\psi(x,t) = c^2(x)\cdot\rho(x)\nabla\cdot\left(\frac{1}{\rho(x)}\nabla\psi(x,t)\right), \tag{1}$$

where $\psi(x,t)$ is the value of the scalar field at a point $x$ at time $t, c(x)$ is the local speed of wave propagation at a point $x$, and $\rho(x)$ is the density at a point $x$. In order to solve (1), we assume

$$\psi(x,t) = \psi_k(x)e^{ikc_0t}, \tag{2}$$

where $k$ is a complex number with non-negative imaginary part, and $c_0$ is the speed of wave propagation outside of the scattering structure. Substituting (2) into (1), we obtain

$$\rho(x)\nabla\cdot\left(\frac{1}{\rho(x)}\nabla\psi_k(x)\right) + k^2\cdot\frac{c_0^2}{c^2(x)}\psi_k(x) = 0. \tag{3}$$

Since the inverse scattering scheme assumes that the speed $c(x,y,z)$ of propagation of sound, the density $\rho(x,y,z)$ and the attenuation $\gamma(x,y,z)$ are independent of the variables $x,y$, i.e., $c(x,y,z) = c(z),\ \rho(x,y,z) = \rho(z),\ \gamma(x,y,z) = \gamma(z)$, (3) can be rewritten by the formula

$$\frac{\partial^2\psi_k}{\partial x^2} + \frac{\partial^2\psi_k}{\partial z^2} - \frac{1}{\rho(z)}\frac{d\rho}{dz}\cdot\frac{\partial\psi_k}{\partial z} + k^2\cdot\frac{c_0^2}{c^2(z)}\cdot\psi_k = 0. \tag{4}$$

Throughout this paper, we use the notation

$$\frac{c_0^2}{c^2(z)} = 1 + q(z) + i\cdot\gamma(z), \tag{5}$$

where $q(z)$ and $\gamma(z)$ are known as potential and attenuation of the layered acoustic medium, and that $\rho, q, \gamma \in c_0^2([0,1])$, i.e., $\rho, q, \gamma$ are twice continuously differentiable everywhere, and are defined by the formulae

$$\rho(x) = \rho(0) = \rho_1, \quad \text{for all } x \leqslant 0, \tag{6}$$
$$\rho(x) = \rho(1) = \rho_2, \quad \text{for all } x \geqslant 1, \tag{7}$$
$$q(x) = q(0) = q_1, \quad \text{for all } x \leqslant 0, \tag{8}$$
$$q(x) = q(1) = q_2, \quad \text{for all } x \geqslant 1, \tag{9}$$
$$\gamma(x) = \gamma(0) = \gamma_1, \quad \text{for all } x \leqslant 0, \tag{10}$$
$$\gamma(x) = \gamma(1) = \gamma_2, \quad \text{for all } x \geqslant 1. \tag{11}$$

Suppose now that the angle of incidence with respect to the normal to the $x - y$ plane is $\theta$, and

$$\psi_k(x,z) = e^{ikx sin\theta}\cdot\phi(z). \tag{12}$$

Substituting (12) into (4), we obtain

$$\phi''(x,k) - \frac{\rho'(x)}{\rho(x)}\cdot\phi'(x,k) + k^2\cdot(1 + q(x) + i\cdot\gamma(x) - \alpha^2)\cdot\phi(x,k) = 0, \tag{13}$$

where

$$\alpha = sin(\theta). \tag{14}$$

Eq. (13) is the well-known scalar Helmholtz equation in a layered acoustic medium. For any complex $k$, we consider solutions of the Helmholtz equation $\phi_+(x,k)$ and $\phi_-(x,k)$ defined by the formulae

$$\phi_+(x,k) = \phi_{\text{inc}+}(x,k) + \phi_{\text{scat}+}(x,k), \tag{15}$$
$$\phi_-(x,k) = \phi_{\text{inc}-}(x,k) + \phi_{\text{scat}-}(x,k) \tag{16}$$

with

$$\phi_{\text{inc}+}(x,k) = A_+(k)e^{ik\sqrt{1+q_1+i\gamma_1-\alpha^2}x} \tag{17}$$

for all $x < 1$,

$$\phi_{\text{inc}+}(x,k) = A_+(k)e^{ik\sqrt{1+q_2+i\gamma_2-\alpha^2}x} \tag{18}$$

for all $x \geqslant 1$,

$$\phi_{\text{inc}-}(x,k) = A_-(k)e^{-ik\sqrt{1+q_1+i\gamma_1-\alpha^2}x} \tag{19}$$

for all $x \leqslant 0$,

$$\phi_{\text{inc}-}(x,k) = A_-(k)e^{-ik\sqrt{1+q_2+i\gamma_2-\alpha^2}x} \tag{20}$$

for all $x > 0$, where $A_\pm(k)$ are complex, and $\phi_{\text{scat}+}, \phi_{\text{scat}-}$ satisfying the outgoing radiation boundary conditions

$$\phi'_{\text{scat}\pm}(0,k) + ik\sqrt{1+q_1+i\gamma_1-\alpha^2}\phi_{\text{scat}\pm}(0,k) = 0, \tag{21}$$

$$\phi'_{\text{scat}\pm}(1,k) - ik\sqrt{1+q_2+i\gamma_2-\alpha^2}\phi_{\text{scat}\pm}(1,k) = 0. \tag{22}$$

Combining Eqs. (13)–(20), we obtain the equations

$$\begin{aligned}&\phi''_{\text{scat}+}(x,k) - \frac{\rho'(x)}{\rho(x)}\cdot\phi'_{\text{scat}+}(x,k) + k^2\cdot(1+q(x)+i\cdot\gamma(x)-\alpha^2)\cdot\phi_{\text{scat}+}(x,k)\\ &\quad = -(k^2((q-q_1)+i(\gamma-\gamma_1)) - \frac{\rho'(x)}{\rho(x)}ik\sqrt{1+q_1+i\gamma_1-\alpha^2})\cdot A_+(k)e^{ik\sqrt{1+q_1+i\gamma_1-\alpha^2}x}\end{aligned} \tag{23}$$

for all $x < 1$,

$$\begin{aligned}&\phi''_{\text{scat}+}(x,k) - \frac{\rho'(x)}{\rho(x)}\cdot\phi'_{\text{scat}+}(x,k) + k^2\cdot(1+q(x)+i\cdot\gamma(x)-\alpha^2)\cdot\phi_{\text{scat}+}(x,k)\\ &\quad = -(k^2((q-q_2)+i(\gamma-\gamma_2)) - \frac{\rho'(x)}{\rho(x)}ik\sqrt{1+q_2+i\gamma_2-\alpha^2})\cdot A_+(k)e^{ik\sqrt{1+q_2+i\gamma_2-\alpha^2}x}\end{aligned} \tag{24}$$

for all $x \geqslant 1$,

$$\begin{aligned}&\phi''_{\text{scat}-}(x,k) - \frac{\rho'(x)}{\rho(x)}\cdot\phi'_{\text{scat}-}(x,k) + k^2\cdot(1+q(x)+i\cdot\gamma(x)-\alpha^2)\cdot\phi_{\text{scat}-}(x,k)\\ &\quad = -(k^2((q-q_1)+i(\gamma-\gamma_1)) + \frac{\rho'(x)}{\rho(x)}ik\sqrt{1+q_1+i\gamma_1-\alpha^2})\cdot A_-(k)e^{-ik\sqrt{1+q_1+i\gamma_1-\alpha^2}x}\end{aligned} \tag{25}$$

for all $x \leqslant 0$,

$$\begin{aligned}&\phi''_{\text{scat}-}(x,k) - \frac{\rho'(x)}{\rho(x)}\cdot\phi'_{\text{scat}-}(x,k) + k^2\cdot(1+q(x)+i\cdot\gamma(x)-\alpha^2)\cdot\phi_{\text{scat}-}(x,k)\\ &\quad = -(k^2((q-q_2)+i(\gamma-\gamma_2)) + \frac{\rho'(x)}{\rho(x)}ik\sqrt{1+q_2+i\gamma_2-\alpha^2})\cdot A_-(k)e^{-ik\sqrt{1+q_2+i\gamma_2-\alpha^2}x}\end{aligned} \tag{26}$$

for all $x > 0$.

Combining Eqs. (23)–(26) with (6)–(11), we observe that, for any complex $k$, there exist complex numbers $\mu_{1\pm}(k)$ and $\mu_{0\pm}(k)$ such that

$$\phi_{\text{scat}\pm}(x,k) = \mu_{1\pm}(k)\cdot A_\pm(k)e^{ik\sqrt{1+q_2+i\gamma_2-\alpha^2}x} \quad \text{for all } x \geqslant 1, \tag{27}$$

$$\phi_{\text{scat}\pm}(x,k) = \mu_{0\pm}(k)\cdot A_\pm(k)e^{-ik\sqrt{1+q_1+i\gamma_1-\alpha^2}x} \quad \text{for all } x \leqslant 0, \tag{28}$$

combining (18), (19), (27), and (28), we obtain

$$\phi_+(x,k) = (1+\mu_{1+}(k))\cdot A_+(k)e^{ik\sqrt{1+q_2+i\gamma_2-\alpha^2}x} \quad \text{for all } x \geqslant 1, \tag{29}$$

$$\phi_-(x,k) = (1+\mu_{0-}(k))\cdot A_-(k)e^{-ik\sqrt{1+q_1+i\gamma_1-\alpha^2}x} \quad \text{for all } x \leqslant 0. \tag{30}$$

Thus, for any complex $k$, the boundary value problems for $\phi_+, \phi_-$ (Eqs. (13)–(22)) are reformulated as initial value problems (Eqs. (13), (29), and (30)). Furthermore, for any $k \in C$, coefficients $1+\mu_{1+}(k)$ and $1+\mu_{0-}(k)$ in (29) and (30) are both nonzero.

### 2.2. The impedance functions

We will be denoting the upper half complex plane by $C^+$. For any $k \in C^+$, we define the functions $p_+, p_- : (R, C^+) \to C$ by the formulae

$$p_+(x,k) = \frac{\phi'_+(x,k)}{ik\rho(x)\phi_+(x,k)}, \tag{31}$$

$$p_-(x,k) = \frac{\phi'_-(x,k)}{-ik\rho(x)\phi_-(x,k)}, \tag{32}$$

where $\phi_+, \phi_-$ are solutions of the Helmholtz equation (13), and $\rho$ is the density of the scattering structure. In a slight generalization of standard definitions (see, e.g., [3,12]), we will be referring to the definitions $p_+, p_-$ as impedance functions.

**Remark 2.1.** It is easy to see that the impedance functions $p_+, p_-$ do not depend on the coefficients $A_\pm(k)$ in initial conditions (29) and (30). Therefore, if we choose $A_+(k) = \frac{1}{1+\mu_{1+}(k)}$, $A_-(k) = \frac{1}{1+\mu_{0-}(k)}$, the initial conditions (29) and (30) become

$$\phi_+(x,k) = e^{ik\sqrt{1+q_2+i\gamma_2-\alpha^2}x} \quad \text{for all } x \geqslant 1, \tag{33}$$

$$\phi_-(x,k) = e^{-ik\sqrt{1+q_1+i\gamma_1-\alpha^2}x} \quad \text{for all } x \leqslant 0. \tag{34}$$

Thus, we formulate the inverse scattering problem for Eq. (13) as follows:*Suppose that each of the functions $\rho, q, \gamma : [0,1] \to R^1$ has m continuous derivatives with $m \geqslant 2$, and that*

$$\rho(x) > 0, \tag{35}$$

$$1 + q(x) - \alpha^2 > 0, \tag{36}$$

$$\gamma(x) > 0, \tag{37}$$

*for all $x \in [0,1]$. Suppose further that we are given the values of the corresponding impedance functions $p_+(0,k)$ for an appropriately chosen collection of real frequencies $\{k_j\}$ and incidence angles $\{\theta_i\}$, and the "initial" values $\rho(0), q(0), \gamma(0)$. We would like to reconstruct $\rho(x), q(x), \gamma(x)$ for all $x \in [0,1]$.*

This paper is devoted to the construction of an algorithm for the solution of the above problem.

## 3. Analytical preliminaries

In this section, we summarize several well-known mathematical facts to be used in the sections below. These facts are given without proofs, since all of them follow easily from the apparatus built in [1,3–5,13].

### 3.1. Notation

In this paper, we will be denoting the upper half plane by $C^+$. We will be denoting by $c_0^m[a,b]$, the space of all functions $R^1 \to R^1$ that have $m$ continuous derivatives ($m \geqslant 2$), and such that $f(x) = f(a)$ for all $x \leqslant a$, and $f(x) = f(b)$ for all $x \geqslant b$. In other words,

$$f \in c_0^m[a,b] \tag{38}$$

means that $f$ has $m$ continuous derivatives and is constant outside the interval $[a,b]$. Further, we denote $f(a)$ by $f_1$, and $f(b)$ by $f_2$.

For any $a > 0$, the region $K(a) \subset C$ is defined by the formula

$$K(a) = \{k | k \in C, \mathrm{Im}(k) \geqslant 0, |k| \geqslant a\}. \tag{39}$$

In other words, $K(a)$ consists of all points $z$ in $C^+$ such that $|z| \geqslant a$.

### 3.2. Basic lemmas

In this section, we introduce several basic lemmas to be used in the sections below following closely to [3].

**Lemma 3.1.** *Suppose that $f \in c^l(R)$ with l a non-negative integer. Suppose further that $f^{(j)}(0) = 0$ for $0 \leqslant j \leqslant l, f^{(l)}$ is absolutely continuous. Then there exists a positive number c such that*

$$\int_0^x f(t)e^{ik(x-t)}\,dt = -\sum_{j=1}^{l}\left(\frac{1}{2ik}\right)^j f^{j-1}(x) + \left(\frac{1}{2ik}\right)^{l+1}(f^l(x) + b(x,k)) \tag{40}$$

*with $b : R \times C^+ \to C$ an absolutely continuous function of $x \in [0,1]$ such that*

$$|b(x,k)| \leqslant c \tag{41}$$

*for all* $x \in [0,1]$, $k \in C^+$. *Furthermore, if* $f(x) = 0$ *for all* $x \geqslant D$ *with D a positive number, then*

$$|b(x,k)| \leqslant c \tag{42}$$

*for all* $(x,k) \in R \times C^+$.

**Lemma 3.2.** *Suppose that* $a : C \to C$ *is an entire function and that* $A : R \times C \to C^{n\times n}$ *is an* $n \times n$*-matrix whose entries* $a_{i,j}(x,z), i,j = 1,\dots,n$ *are continuous functions of x and entire functions of z for all* $x \in R$ *Then for any* $z \in C$, *the differential equation*

$$y'(x,z) = A(x,z)\cdot y(x,z) \tag{43}$$

*subject to* the initial condition

$$y(0) = c(z) \tag{44}$$

*has an unique solution* $y(x,z)$ *for all* $x \in R$. *Moreover,* $y(x,z)$ *is an entire function of z.*

### 3.3. Schrödinger equation and Riccati equation

This section describes the basic facts about the Helmholtz equation and its connections with the Schrödinger equation and the Riccati equation in the context of scattering problems. Lemma 3.3 describes the fact that a Schrödinger equation with outgoing radiation conditions can be converted into a second kind integral equation with the Green's function of the corresponding Helmholtz equation. Lemma 3.4 describes the Green's function for Helmholtz equation with outgoing radiation conditions.

**Lemma 3.3.** *Suppose that* $G_k : [0,1] \times [0,1] \to C$ *is the Green's function of the boundary value problem*

$$\psi''(x,k) + k^2\psi(x,k) = 0 \tag{45}$$
$$\psi'(0,k) + ik\psi(0,k) = 0 \tag{46}$$
$$\psi'(1,k) - ik\psi(1,k) = 0 \tag{47}$$

*for any complex* $k \neq 0$. *Then the boundary value problem*

$$\psi''(x,k) + (k^2 + \eta(x))\psi(x,k) = f(x,k) \tag{48}$$
$$\psi'(0,k) + ik\psi(0,k) = 0 \tag{49}$$
$$\psi'(1,k) - ik\psi(1,k) = 0 \tag{50}$$

*is equivalent to a second kind integral equation*

$$\psi(x,k) = -\int_0^1 G_k(x,t)\eta(t)\psi(t,k)dt + g(x,k) \tag{51}$$

*with* $f,g : [0,1] \times C \to C$ *and g defined by the formula*

$$g(x,k) = \int_0^1 G_k(x,t)f(t,k)dt. \tag{52}$$

**Lemma 3.4.** *For any complex* $k \neq 0$, *the Helmholtz equation*

$$\psi''(x,k) + k^2\psi(x,k) = 0 \tag{53}$$

*with the outgoing radiation conditions (21) and (22) has the Green's function*

$$G_k(x,t) = \frac{1}{2ik}\begin{cases} e^{ik(t-x)} & x \leqslant t, \\ e^{ik(x-t)} & x \geqslant t. \end{cases} \tag{54}$$

The following lemma connects the solutions of the Helmholtz equation to those of Schrödinger equation via direct transform.

**Lemma 3.5.** *Suppose that* $q,\gamma,\rho : R \to R$ *are* $c^2$*-functions such that* $1 + q(x) - \alpha^2 > 0, \gamma(x) > 0, \rho(x) > 0$, *for all* $x \in R$, *functions* $n,t : R \to C$ *are defined by the formulae*

$$n(x) = \sqrt{1 + q(x) + i\gamma(x) - \alpha^2}, \tag{55}$$
$$t(x) = \int_0^x n(\tau)\,d\tau, \tag{56}$$

*and that contour $\Gamma$ is the image of the mapping $R^1 \to C^1$ defined by the formula*

$$t(x) = \int_0^x \sqrt{1 + q(\tau) - \alpha^2 + i\gamma(\tau)}\, d\tau. \tag{57}$$

*Suppose further that the functions $\eta, g : \Gamma \to C$ are defined by the formulae*

$$\eta(t) = \frac{1}{4}(1 + q(x) + i\gamma(x) - \alpha^2)^{-3} \times \left( \left( 2\frac{\rho''(x)}{\rho(x)} - 3\left(\frac{\rho'(x)}{\rho(x)}\right)^2 \right) \cdot (1 + q(x) + i\gamma(x) - \alpha^2)^2 - (q''(x) + i\gamma''(x)) \cdot (1 + q(x) + i\gamma(x) - \alpha^2) + \frac{5}{4}(q'(x) + i\gamma'(x))^2 \right), \tag{58}$$

$$g(t) = f(x) \cdot \rho^{-\frac{1}{2}}(x) \cdot (1 + q(x) + i\gamma(x) - \alpha^2)^{-\frac{3}{4}}. \tag{59}$$

*Finally, suppose that the function $\phi : R \times C \to C$ satisfies the equation*

$$\phi''(x,k) - \frac{\rho'(x)}{\rho(x)}\phi'(x,k) + k^2(1 + q(x) + i\gamma - \alpha^2)\phi(x,k) = f(x), \tag{60}$$

*and the function $\psi : \Gamma \times C \to C$ is defined by the formula*

$$\psi(t,k) = \rho^{-\frac{1}{2}}(x) \cdot (1 + q(x) + i\gamma(x) - \alpha^2)^{\frac{1}{4}} \cdot \phi(x,k). \tag{61}$$

*Then $\psi$ satisfies the Schrödinger equation*

$$\psi''(t,k) + (k^2 + \eta(t)) \cdot \psi(t,k) = g(t). \tag{62}$$

**Corollary 3.6.** *Suppose that under the conditions of the preceding lemma, $q, \gamma, \rho \in c_0^2([0,1])$. Suppose further that the functions $\psi_+, \psi_- : \Gamma \times C \to C$ are defined by the formulae*

$$\psi_+(t,k) = \rho^{-\frac{1}{2}}(x) \cdot (1 + q(x) + i\gamma(x) - \alpha^2)^{\frac{1}{4}} \cdot \phi_+(x,k), \tag{63}$$

$$\psi_-(t,k) = \rho^{-\frac{1}{2}}(x) \cdot (1 + q(x) + i\gamma(x) - \alpha^2)^{\frac{1}{4}} \cdot \phi_-(x,k). \tag{64}$$

*Then $\psi_+, \psi_-$ satisfy the ODEs*

$$\psi_+''(t,k) + (k^2 + \eta(t)) \cdot \psi_+(t,k) = 0, \tag{65}$$

$$\psi_-''(t,k) + (k^2 + \eta(t)) \cdot \psi_-(t,k) = 0 \tag{66}$$

*subject to the boundary conditions*

$$\psi_+(t,k) = \xi(k) \cdot e^{ik(t-T_1)} \tag{67}$$

*for all* $\mathrm{Re}(t) \geqslant \mathrm{Re}(T_1)$, *and*

$$\psi_-(t,k) = \rho_1^{-\frac{1}{2}}(1 + q_1 - \alpha^2 + i\gamma_1)^{\frac{1}{4}} e^{-ikt}, \tag{68}$$

*for all* $\mathrm{Re}(t) \leqslant 0$ *with* $T_1, \xi(k) \neq 0$ *defined by the formulae*

$$T_1 = \int_0^1 \sqrt{1 + q(x) + i\gamma(x) - \alpha^2}\, dx, \tag{69}$$

$$\xi(k) = \rho_2^{-\frac{1}{2}}(1 + q_2 - \alpha^2 + i\gamma_2)^{\frac{1}{4}} e^{ik\sqrt{1+q_2+i\gamma_2-\alpha^2}}. \tag{70}$$

*Furthermore,*

$$p_+(x,k) = \sqrt{1 + q(x) + i\gamma(x) - \alpha^2}\,\frac{\psi_+'(t,k)}{ik\rho(x)\psi_+(t,k)} + \frac{\frac{1}{2}\frac{\rho'(x)}{\rho(x)} - \frac{1}{4}(1 + q(x) + i\gamma(x) - \alpha^2)^{-1} \cdot (q'(x) + i\gamma'(x))}{ik\rho(x)}, \tag{71}$$

$$p_-(x,k) = \sqrt{1 + q(x) + i\gamma(x) - \alpha^2}\,\frac{\psi_-'(t,k)}{-ik\rho(x)\psi_-(t,k)} - \frac{\frac{1}{2}\frac{\rho'(x)}{\rho(x)} - \frac{1}{4}(1 + q(x) + i\gamma(x) - \alpha^2)^{-1} \cdot (q'(x) + i\gamma'(x))}{ik\rho(x)}. \tag{72}$$

**Remark 3.1.** Lemma 3.5 provides a connection between the solutions of the Helmholtz equation (60) and those of the appropriately chosen Schrödinger equation (62). This connection will be used in the following chapter as an analytical tool. However, it is not useful in numerical computations since the connection between $\eta$ and $q$ (see Eq. (58)) is generally ill-conditioned.

**Observation 3.2.** Suppose that $q, \gamma, \rho \in c_0^2([0,1])$. Then according to Lemma 3.5 and Corollary 3.6,

$$t = \sqrt{1 + q_1 + i\gamma_1 - \alpha^2}x, \tag{73}$$

and consequently

$$\phi_+(x,k) = \sqrt{\rho_1} \cdot (1 + q_1 + i\gamma_1 - \alpha^2)^{-\frac{1}{4}} \cdot \psi_+(t,k), \tag{74}$$

for all $x \leqslant 0$. Now, suppose the function $\psi_+$ is defined by formula (63). Defining the scattered field $\psi_{\text{scat}+} : \Gamma \times C \to C$ by the formula

$$\psi_+(t,k) = \rho_1^{-\frac{1}{2}} \cdot \left(\sqrt{1 + q_1 + i\gamma_1 - \alpha^2}\right)^{\frac{1}{4}} \cdot (e^{ikt} + \psi_{\text{scat}+}(t,k)), \tag{75}$$

we immediately obtain the Schrödinger equation

$$\psi''_{\text{scat}+}(t,k) + (k^2 + \eta(t))\psi_{\text{scat}+}(t,k) = \rho(x)^{-\frac{1}{2}} \cdot (1 + q(x) + i\gamma(x) - \alpha^2)^{-\frac{3}{4}} \cdot \left(-k^2((q - q_1) + i(\gamma - \gamma_1)) + \frac{\rho'(x)}{\rho(x)} ik \times \sqrt{1 + q_1 + i\gamma_1 - \alpha^2}\right) \cdot e^{ik\sqrt{1+q_1+i\gamma_1-\alpha^2}x} \tag{76}$$

subject to outgoing radiation conditions (21) and (22).

The following lemma introduces the Riccati equations satisfied by the impedance functions $p_+, p_-$. They are obtained by substituting (31) and (32) into the Helmholtz equation (13).

**Lemma 3.7.** *Suppose that under the conditions of the preceding lemma,*

$$\psi_+(x_0,k_0) \neq 0, \tag{77}$$
$$\psi_-(x_0,k_0) \neq 0, \tag{78}$$

*at some point $(x_0,k_0) \in R \times C$. Then there exists a neighborhood $D$ of $(x_0,k_0)$ such that the impedance functions $p_+, p_-$ satisfy the Riccati equations*

$$p'_+(x,k) = -ik\rho(x) \cdot \left(p_+^2(x,k) - \frac{1 + q(x) + i\gamma(x) - \alpha^2}{\rho^2(x)}\right), \tag{79}$$

$$p'_-(x,k) = ik\rho(x) \cdot \left(p_-^2(x,k) - \frac{1 + q(x) + i\gamma(x) - \alpha^2}{\rho^2(x)}\right), \tag{80}$$

*for all $(x,k) \in D$.*

**Observation 3.3.** Combining formulae (33) and (34), we easily observe that

$$p_+(x,k) = \frac{\sqrt{1 + q_2 + i\gamma_2 - \alpha^2}}{\rho_2}, \quad \text{for all } x \geqslant 1 \tag{81}$$

$$p_-(x,k) = \frac{\sqrt{1 + q_1 + i\gamma_1 - \alpha^2}}{\rho_1}, \quad \text{for all } x \leqslant 0 \tag{82}$$

for all complex $k \neq 0$.

**Observation 3.4.** If $\gamma(x) = 0$ for all $x \in R$, it is easy to see from Eqs. (79)–(82) that

$$\overline{p_+(x,k)} = p_+(x,-k), \tag{83}$$
$$\overline{p_-(x,k)} = p_-(x,-k), \tag{84}$$

for all real $k$, since $\overline{p_+(x,k)}$ and $p_+(x,-k)$ satisfy identical differential equations and boundary conditions, and the same is true for $\overline{p_-(x,k)}$ and $p_-(x,-k)$, too.

## 4. Mathematical apparatus

In this section, we introduce analytical tools to be used in the construction of the algorithms of this paper. This section discusses the following three facts.

(A) For any $x \in R$, the impedance functions $p_+(x,k), p_-(x,k)$, defined by (31) and (32), are analytic functions of $k$ in the upper half plane $C^+$. Furthermore,

$$p_+(x,k) = \frac{1}{\rho(x)} \cdot \sqrt{1+q(x)+i\gamma-\alpha^2} - \frac{1}{ik} \cdot \frac{\rho(x)\cdot(q'(x)+i\gamma'(x)) - 2\cdot(1+q(x)+i\gamma(x)-\alpha^2)\cdot\rho'(x)}{4\cdot\rho^2(x)\cdot(1+q(x)+i\gamma(x)-\alpha^2)} + O(k^{-2}), \tag{85}$$

$$p_-(x,k) = \frac{1}{\rho(x)} \cdot \sqrt{1+q(x)+i\gamma-\alpha^2} + \frac{1}{ik} \cdot \frac{\rho(x)\cdot(q'(x)+i\gamma'(x)) - 2\cdot(1+q(x)+i\gamma(x)-\alpha^2)\cdot\rho'(x)}{4\cdot\rho^2(x)\cdot(1+q(x)+i\gamma(x)-\alpha^2)} + O(k^{-2}) \tag{86}$$

as $|k| \to \infty$ for all $x \in R$, $k \in C^+$.

(B) For any fixed $x \in R$, the difference between the impedance functions $p_+(x,-k)$ and $p_-(x,k)$ decays like a constant times $k^{-m}$, where $k \in R$, and $m$ is the smoothness of the scatterer. In other words,

$$p_+(x,-k) - p_-(x,k) = O(k^{-m}), \tag{87}$$

as $|k| \to \infty$, $k \in R$.

(C) For any $a > 0$, and all $x \in R$,

$$\begin{aligned} &\rho(x)\cdot(q'(x)+i\gamma'(x)) - 2\cdot\rho'(x)\cdot(1+q(x)+i\gamma(x)-\alpha^2) \\ &\quad = \frac{2}{\pi}(1+q(x)+i\gamma(x)-\alpha^2)\rho^2(x)\int_{-a}^{a}(p_+(x,k)-p_-(x,k))\,dk + O(a^{-(m-1)}), \end{aligned} \tag{88}$$

where $m$ is the smoothness of the scatterer, $p_+(x,k)$ and $p_-(x,k)$ are impedance functions defined by (31) and (32), $\rho$ is the density of the scattering object, $q$ is the potential, and $\gamma$ is the attenuation. (88) is an example of a trace formula.The proofs in this section are modeled after those in [3], and details can be found in [7].

### 4.1. Boundedness

This section establishes the basic properties of the impedance functions $p_+, p_-$, defined by (31) and (32). Lemma 4.1 describes the behavior of $\phi_+, \phi_-$ in the vicinity of $k = 0$ in the complex plane. Lemma 4.2 describes the properties of the impedance functions $p_+, p_-$ near $k = 0$. Lemma 4.3 shows that $\phi_-, \phi_+$ are nonzero for all real $x$ and complex $k \neq 0$.

The following lemma describes the behavior of $\phi_+, \phi_-$ in the vicinity of $k = 0$ in the complex plane.

**Lemma 4.1.** *Suppose that $\rho, q, \gamma \in c_0^2([0,1])$, and $A > 0$ is a real number. Then, there exist positive numbers $\delta, \varepsilon$, and $\eta$ such that*

$$|\phi_+(x,k) - 1| \leqslant \varepsilon|k|, \tag{89}$$

$$|\phi_-(x,k) - 1| \leqslant \varepsilon|k|, \tag{90}$$

$$\left|\phi'_+(x,k) - ik\sqrt{1+q_2+i\gamma_2-\alpha^2}\cdot\frac{\rho(x)}{\rho_2}\right| \leqslant \eta|k|^2, \tag{91}$$

$$\left|\phi'_-(x,k) + ik\sqrt{1+q_1+i\gamma_1-\alpha^2}\cdot\frac{\rho(x)}{\rho_1}\right| \leqslant \eta|k|^2, \tag{92}$$

$$\phi_+(x,k) \neq 0, \tag{93}$$

$$\phi_-(x,k) \neq 0 \tag{94}$$

*for all real $x \in [-A, A]$ and complex $k$ such that $|k| < \delta$. In (89)–(94), $q_1, q_2, \gamma_1, \gamma_2$, are defined in Section 3.1, $\alpha$ is define by (14), and $\phi_\pm(x,k)$ is the field at $x$.*

The following lemma describes the properties of the impedance functions $p_+, p_-$ near $k = 0$.

**Lemma 4.2.** *Suppose that $\rho, q, \gamma \in c_0^2([0,1])$ and $A > 0$ is a real number. Then there exists a real number $\delta > 0$ such that the impedance functions $p_+, p_-$, defined by (31) and (32), are continuous functions of $(x,k)$ for all real $(x,k) \in D$, where $D$ is the set of all pairs $(x,k)$, where $x \in [-A, A], k \in C, k \neq 0, |k| \neq \delta$. Furthermore,*

$$\lim_{k\to 0} p_+(x,k) = \frac{\sqrt{1+q_2+i\gamma_2-\alpha^2}}{\rho_2}, \tag{95}$$

$$\lim_{k\to 0} p_-(x,k) = \frac{\sqrt{1+q_1+i\gamma_1-\alpha^2}}{\rho_1}, \tag{96}$$

*where $q_1, q_2, \gamma_1, \gamma_2, \rho_1, \rho_2$ are defined in Section 3.1, $\alpha, p_+, p_-$ are defined by (14), (31), and (32),* respectively.

**Proof.** Due to Lemma 4.1, there exists a real number $\delta > 0$ such that $\phi_+(x,k) \neq 0, \phi_-(x,k) \neq 0$ for all real $(x,k) \in D$. Therefore, the impedance functions $p_+, p_-$ are well-defined in $D$, and their continuity follows from the continuity of $\phi_+, \phi_-, \phi'_+, \phi'_+, \rho$, as well as their definitions (31) and (32). Eqs. (95) and (96) are obtained via the direct application of (89)–(92) and (31) and (32). □

**Remark 4.1.** Due to Lemma 4.2, if we define

$$p_+(x,0) = \frac{\sqrt{1+q_2+i\gamma_2-\alpha^2}}{\rho_2}, \tag{97}$$

$$p_-(x,0) = \frac{\sqrt{1+q_1+i\gamma_1-\alpha^2}}{\rho_1}, \tag{98}$$

then $p_+, p_-$ are continuous functions at $k = 0$.

The following lemma states that $\phi_+, \phi_-, \phi'_+, \phi'_-$ are nonzero for all real $x$ and complex $k \neq 0$.

**Lemma 4.3.** *For all $x \in R$ and complex $k \neq 0$,*

$$\phi_+(x,k) \neq 0, \tag{99}$$

$$\phi'_+(x,k) \neq 0, \tag{100}$$

$$\phi_-(x,k) \neq 0, \tag{101}$$

$$\phi'_-(x,k) \neq 0. \tag{102}$$

**Proof.** Since the proofs of this lemma for $\phi_+, \phi'_+$ and for $\phi_-, \phi'_-$ are nearly identical, we only prove (101) and (102). Since $\phi_-$ satisfies (13) with the boundary condition (34), we decompose $\phi_-$ into two parts via the formulae

$$\phi_-(x,k) = u(x,k) + iv(x,k), \tag{103}$$

$$\phi'_-(x,k) = u'(x,k) + iv'(x,k), \tag{104}$$

where functions $u, v$: $R \times C \to C$ satisfy Eq. (13) with boundary conditions

$$u(x,k) = \cos\left(k\sqrt{1+q_1+i\gamma_1-\alpha^2}x\right), \tag{105}$$

$$v(x,k) = \sin\left(k\sqrt{1+q_1+i\gamma_1-\alpha^2}x\right) \tag{106}$$

for all $x \leqslant 0$ and $k \neq 0$. The Wronskian $W(u, v)$ of the pair $u, v$ is

$$W(u,v) = \sqrt{1+q_1+i\gamma_1-\alpha^2} \cdot k \tag{107}$$

for any $x \in R$. Therefore, for any $k \neq 0, u(x,k), v(x,k)$ can not be zero simultaneously, nor can $u'(x,k), v'(x,k)$. Now, (101) and (102) follows immediately given (103) and (104). □

### 4.2. Asymptotics and smoothness

The principal purpose of this section is to formulate and prove the facts (A) and (B) described in the beginning of Section 4. We begin by deriving Eqs. (85) and (86), and prove Lemma 4.4, under the assumption that such asymptotic forms exist for impedance functions. Then, we demonstrate the existence of such asymptotic expansions (Lemma 4.9), by converting the Schrödinger equation into an integral equation (Lemma 4.5) and applying the Neumann series (Lemma 4.6). Finally, the statements (A) and (B) are proved in Theorems 4.3 and 4.4.

The following lemma yields the first two terms in the asymptotic expansions of the impedance functions $p_-, p_+$.

**Lemma 4.4.** *Suppose that impedance functions $p_+(x,k)$ and $p_-(x,k)$ are defined by (31) and (32), and that*

$$p_+(x,k) = a_0(x) + \frac{a_1(x)}{ik} + \frac{a_2(x)}{(ik)^2} + \cdots + \frac{a_{m-1}(x)}{(ik)^{m-1}} + O(k^{-m}), \tag{108}$$

$$p_-(x,k) = b_0(x) + \frac{b_1(x)}{ik} + \frac{b_2(x)}{(ik)^2} + \cdots + \frac{b_{m-1}(x)}{(ik)^{m-1}} + O(k^{-m}) \tag{109}$$

*for large real $k$, and integer $m \geqslant 2$. Then,*

$$a_0(x) = b_0(x) = \frac{1}{\rho(x)} \cdot \sqrt{1+q(x)+i\gamma-\alpha^2}, \tag{110}$$

*and*

$$a_1(x) = -b_1(x) = -\frac{\rho(x)\cdot(q'(x)+i\gamma'(x)) - 2\cdot(1+q(x)+i\gamma(x)-\alpha^2)\cdot\rho'(x)}{4\cdot\rho^2(x)\cdot(1+q(x)+i\gamma(x)-\alpha^2)}, \tag{111}$$

*where $\rho, q, \gamma, \alpha$ are defined in (13).*

**Proof.** It is easily observed from (79) and (80) that the impedance functions $p_+(x,-k), p_-(x,k)$ satisfy the same Riccati differential Eq. (80). Hence,

$$a_i(x) = b_i(x), \quad \text{for all the even } i, \tag{112}$$
$$a_i(x) = -b_i(x), \quad \text{for all the odd } i. \tag{113}$$

Identities (110) and (111) are obtained by substituting (109) into (80) and comparing terms at the appropriate powers of $k$. □

The following lemma converts the Schrödinger equation (66) into an integral equation.

**Lemma 4.5.** *Suppose that $\psi_-$ is defined in (64), and $q,\rho,\gamma$ are functions in $c_0^2([0,1])$ such that for all $x\in[0,1], 1+q(x)-\alpha^2>0, \gamma(x)>0, \rho(x)>0$. Suppose further, for all $x\in R$ and complex $k\neq 0$,*

$$t(x) = \int_0^x \sqrt{1+q(\tau)-\alpha^2+i\gamma(\tau)}\,d\tau, \tag{114}$$
$$m(t,k) = e^{ikt}\psi_-(t,k), \tag{115}$$
$$n(t,k) = -\frac{e^{ikt}}{ik}\psi'_-(t,k). \tag{116}$$

*Then,*

$$m = F_k(m) + (1+q_1+i\gamma_1-\alpha^2)^{\frac{1}{4}}\rho_1^{-\frac{1}{2}}, \tag{117}$$
$$n(t,k) = m(t,k) - \frac{1}{2ik}\int_0^t \eta(\tau)e^{2ik(t-\tau)}m(\tau,k)\,d\tau, \tag{118}$$

*where*

$$F_k(f)(t) = \frac{1}{2ik}\int_0^t \eta(\tau)(1-e^{2ik(t-\tau)})f(\tau)\,d\tau. \tag{119}$$

**Proof.** Combining (66) and (68) with (115) and (116), we observe that $m$ satisfies the equation

$$m''(t,k) - 2ikm'(t,k) = -\eta(t)m(t,k) \tag{120}$$

subject to the initial conditions

$$m(0,k) = (1+q_1+i\gamma_1-\alpha^2)^{\frac{1}{4}}\rho_1^{-\frac{1}{2}}, \tag{121}$$
$$m'(0,k) = 0. \tag{122}$$

Multiplying (120) by $e^{-2ikt}$ and integrating the result from 0 to $t$, we have

$$m'(t,k) = -\int_0^t \eta(\tau)e^{2ik(t-\tau)}m(\tau,k)\,d\tau. \tag{123}$$

Now, (117) is obtained immediately via integrating (123) from 0 to $t$, and (118) follows from (123), (117) and (118). □

**Observation 4.2.** Since $\eta(\tau)$ is continuous on the entire complex plane and zero outside of a bounded region, the functions $\eta(\tau)(1-e^{2ik(t-\tau)})$ and $\eta(\tau)e^{2ik(t-\tau)}$ are bounded for all real $t$, $\tau$, and $k\in C^+$. Therefore, there exists a real number $c_1>0$ such that

$$|F_k| \leqslant \frac{c_1}{|k|}, \tag{124}$$

and hence there exists a real number $A>0$ such that

$$|F_k| \leqslant 1 \tag{125}$$

for all $k\in K(A)$ (see (39) for the definition of $K(A)$).

Lemmas 4.6 and 4.7 analyze the Neumann series for the integral Eq. (117).

**Lemma 4.6.** *Suppose that $q,\rho,\gamma$ are three functions in $c_0^{\mu}([0,1])$ with integer $\mu\geqslant 2$. Suppose further that contour $\Gamma$ is the image of the mapping $R^1\to C^1$ defined by the formula*

$$t(x) = \int_0^x \sqrt{1+q(\tau)-\alpha^2+i\gamma(\tau)}\,d\tau. \tag{126}$$

*Then for any integer $1\leqslant l\leqslant\mu$, there exist functions $a_j:\Gamma\to R, j=1,\ldots,\mu-1,$, and $a_\mu:\Gamma\times C^+\to C$, such that*

$$m_l(t,k) = (1+q_1+i\gamma_1-\alpha^2)^{\frac{1}{4}}\rho_1^{-\frac{1}{2}} + \sum_{j=1}^{\mu-1}\left(\frac{1}{2ik}\right)^j a_j(t) + \left(\frac{1}{2ik}\right)^{\mu} a_\mu(t,k), \tag{127}$$

*where $m_l : \Gamma\times C^+ \to C$ is defined by the formulae*

$$m_0(t,k) = 0, \tag{128}$$

$$m_l(t,k) = (1+q_1+i\gamma_1-\alpha^2)^{\frac{1}{4}}\rho_1^{-\frac{1}{2}} + F_k(m_{l-1})(t,k) = (1+q_1+i\gamma_1-\alpha^2)^{\frac{1}{4}}\rho_1^{-\frac{1}{2}} + \frac{1}{2ik}\int_0^t \eta(\tau)(1-e^{2ik(t-\tau)})m_{l-1}(\tau,k)\,d\tau. \tag{129}$$

*In (127), $\frac{d^{\mu-j}a_j(t)}{dt^{\mu-j}}$ are bounded and absolutely continuous for all $t\in\Gamma, j=1,\dots,\mu-1$, and $a_\mu(t,k)$ is bounded and absolutely continuous function of $t$ for all $(t,k)\in\Gamma\times C^+$.*

**Proof.** We prove this lemma by induction. For $l=1$, formulae (128) and (129) yield

$$m_1(t,k) = (1+q_1+i\gamma_1-\alpha^2)^{\frac{1}{4}}\rho_1^{-\frac{1}{2}} \tag{130}$$

for all $(t,k)\in\Gamma\times C^+$, which is already in the form (127). For $l\geqslant 1$, assuming that $m_l(t,k)$ is in the form (127), we obtain $m_{l+1}$ using (129):

$$\begin{aligned} m_{l+1}(t,k) &= (1+q_1+i\gamma_1-\alpha^2)^{\frac{1}{4}}\rho_1^{-\frac{1}{2}} + \frac{1}{2ik}\int_0^t \eta(\tau)(1-e^{2ik(t-\tau)})m_l(\tau,k)\,d\tau \\ &= (1+q_1+i\gamma_1-\alpha^2)^{\frac{1}{4}}\rho_1^{-\frac{1}{2}} + I_1(t,k)+I_2(t,k)+I_3(t,k)+I_4(t,k) \end{aligned} \tag{131}$$

with $I_j : \Gamma\times C^+\to C, 1\leqslant j\leqslant 4$ defined by the formulae

$$I_1(t,k) = \frac{1}{2ik}\int_0^t \eta(\tau)\,d\tau + \sum_{j=2}^{\mu-1}\left(\frac{1}{2ik}\right)^j\int_0^t \eta(\tau)a_{j-1}(\tau)\,d\tau, \tag{132}$$

$$I_2(t,k) = -\frac{1}{2ik}\int_0^t \eta(\tau)(1-e^{2ik(t-\tau)})d\tau, \tag{133}$$

$$I_3(t,k) = -\sum_{s=2}^{\mu-1}\left(\frac{1}{2ik}\right)^s\int_0^t \eta(\tau)a_{s-1}(\tau)e^{2ik(t-\tau)}\,d\tau \equiv -\sum_{s=2}^{\mu-1} J_s(t,k), \tag{134}$$

$$I_4(t,k) = \frac{1}{2ik}\int_0^t \eta(\tau)a_\mu(\tau)(1-e^{2ik(t-\tau)})\,d\tau. \tag{135}$$

Clearly, we only need to show that $I_j, 1\leqslant j\leqslant 4$ can be expressed in the form

$$\sum_{j=1}^{\mu-1}\left(\frac{1}{2ik}\right)^j\alpha_j(t) + \left(\frac{1}{2ik}\right)^{\mu}\alpha_\mu(t,k). \tag{136}$$

Obviously, $I_1$ and $I_4$ are already in the form (136). We now use Lemma 3.1 to show that $I_2, I_3$ can also be expanded in the form (136). Observing that $\eta(t(x)) = 0$ for all $x\notin(0,1), \eta^{(\mu-2)}$ is absolutely continuous, and that $a_j^{(\mu-j)}, 1\leqslant j\leqslant \mu-1$ are absolutely continuous (due to the inductive assumption), we can use formula (40) to expand $I_2$ and each term $J_s(s=1,\dots,\mu-1)$ of $I_3$ as

$$I_2(t,k) = \sum_{j=2}^{\mu-1}\left(\frac{1}{2ik}\right)^j\eta^{(j-2)}(t) + \left(\frac{1}{2ik}\right)^{\mu} b_1(t,k), \tag{137}$$

$$J_s(t,k) = \left(\frac{1}{2ik}\right)^s\int_0^t \eta(\tau)a_{s-1}(\tau)e^{2ik(t-\tau)}\,d\tau = -\sum_{j=s+1}^{\mu-1}\left(\frac{1}{2ik}\right)^j\frac{d^{(j-s-1)}}{dt^{(j-s-1)}}(\eta(t)a_{s-1}(t)) - \left(\frac{1}{2ik}\right)^{\mu} b_s(t,k) \tag{138}$$

with $b_s : \Gamma\times C^+\to C$ uniformly bounded on $\Gamma\times C^+$ (see Lemma 3.1). Therefore, $I_2$ is in the form (136) due to (137), and $I_3$ is of the form (136) due to (138) and (134). Thus, $m_{l+1}(t,k)$ can indeed be written in the form (127). □

**Lemma 4.7.** *Under the conditions of the preceding lemma, suppose that the functions $m, n, m_\mu, n_\mu : \Gamma\times C^+\to C$ are defined by the formulae (117), (118) and (129) and*

$$n_\mu(t,k) = m_\mu(t,k) - \frac{1}{2ik}\int_0^t \eta(\tau)e^{2ik(t-\tau)}m_\mu(\tau,k)\,d\tau, \tag{139}$$

*respectively. Then there exist positive real numbers $A, c_1, c_2, c_3$ such that*

$$|m(t,k) - m_\mu(t,k)| \leqslant \frac{c_1}{|k|^{\mu}}, \tag{140}$$

$$|n(t,k) - n_\mu(t,k)| \leqslant \frac{c_2}{|k|^{\mu}} \tag{141}$$

*for all* $(t,k) \in \Gamma \times K(A)$, *and*

$$|\frac{n(t,k)}{m(t,k)} - 1| \leqslant \frac{c_3}{|k|^{\mu}} \tag{142}$$

*for all* $(t,k) \in [T_1, \infty) \times K(A)$.

**Proof.** Due to (124), the norm of the integral operator $F_k$ in (129) is of the order $O(|k|^{-1})$ for any $k \in C^+$, from which (140) follows immediately.

Subtracting (139) from (118), we obtain

$$n(t,k) - n_{\mu}(t,k) = m(t,k) - m_{\mu}(t,k) - \frac{1}{2ik}\int_0^t \eta(\tau)e^{2ik(t-\tau)}(m(\tau,k) - m_{\mu}(\tau,k))\,d\tau. \tag{143}$$

Now, the estimate (141) is a direct consequence of (143) and (140), and the fact that

$$\frac{1}{2ik}\eta(\tau)e^{2ik(t-\tau)} \tag{144}$$

is bounded for all $k \in K(A), -\infty < \tau \leqslant t < \infty$ (see Observation 4.2). We now prove (142) by showing that there exists a positive number $c_3$ such that

$$\left|\frac{n_{\mu}(t,k)}{m_{\mu}(t,k)} - 1\right| \leqslant \frac{c_3}{|k|^{\mu}} \tag{145}$$

for all $(t,k) \in [T_1, \infty) \times K(A)$. Indeed, Lemma 4.6 states that $a_{\mu}(t,k)$ in (127) is bounded and absolutely continuous for all $(t,k) \in \Gamma \times C^+$, and $a_j(t)$ in (127) is also independent of $k$. Therefore, we can assume that the constant $A$ has been chosen so that for all $(t,k) \in \Gamma \times K(A)$,

$$\left|\sum_{j=1}^{\mu-1}\left(\frac{1}{2ik}\right)^j a_j(t) + \left(\frac{1}{2ik}\right)^{\mu} a_{\mu}(t,k)\right| \leqslant \frac{1}{2}\cdot(1 + q_1 + i\gamma_1 - \alpha^2)^{\frac{1}{4}}\rho_1^{-\frac{1}{2}}, \tag{146}$$

or equivalently,

$$|m_{\mu}(t,k)| \geqslant \frac{1}{2}\cdot(1 + q_1 + i\gamma_1 - \alpha^2)^{\frac{1}{4}}\rho_1^{-\frac{1}{2}}. \tag{147}$$

Combining (139) with (127), we obtain

$$n_{\mu} = m_{\mu} + I_2(t,k) + I_3(t,k) + I_5(t,k), \tag{148}$$

with $I_2(t,k), I_3(t,k)$ defined by (133) and (134), and $I_5(t,k)$ defined by the formula

$$I_5(t,k) = \left(\frac{1}{2ik}\right)^{\mu+1}\int_0^t \eta(\tau)a_{\mu}(\tau,k)e^{2ik(t-\tau)}\,d\tau. \tag{149}$$

Noticing that $\eta(\tau) = 0$ for all $\mathrm{Re}(t) \geqslant \mathrm{Re}(T_1)$, we have

$$I_2(t,k) = \left(\frac{1}{2ik}\right)^{\mu} b_1(t,k), \tag{150}$$

$$J_s(t,k) = \left(\frac{1}{2ik}\right)^{\mu} b_s(t,k) \tag{151}$$

for all $(t,k) \in [T_1, \infty) \times K(A)$, due to (137) and (138). Consequently, there exists $c > 0$ such that

$$|I_2(t,k) + I_3(t,k) + I_5(t,k)| \leqslant \frac{c}{|k|^{\mu}} \tag{152}$$

for all $(t,k) \in [T_1, \infty) \times K(A)$, since $a_{\mu}(t,k), b_s(t,k)$ are bounded for all $(t,k) \in [T_1, \infty) \times K(A)$, and $s = 1, \ldots, \mu - 1$.

Now, the estimate (145) is a direct consequence of (148), (152) and (147). The estimate (142) is a direct consequence of (145), (140), and (141). □

The following lemma is the counterpart of Lemma 4.7 for $\psi_+(t,k)$; the proof is similar and therefore omitted.

**Lemma 4.8.** *Suppose* $\psi_+$ *is defined in (63),* $q, \rho, \gamma$ *are three functions in* $c_0^2([0,1])$ *such that* $1 + q(x) - \alpha^2 > 0, \gamma(x) > 0, \rho(x) > 0$ *for all x. Suppose further, for all* $x \in R$ *and complex* $k \neq 0$,

$$t(x) = \int_0^x \sqrt{1 + q(\tau) - \alpha^2 + i\gamma(\tau)}\,d\tau, \tag{153}$$

$$f(t,k) = e^{-ikt}\psi_+(t,k), \tag{154}$$

$$g(t,k) = \frac{e^{-ikt}}{ik}\psi'_+(t,k). \tag{155}$$

*Then under the conditions of the Lemma 4.7, there exist positive numbers $A, d_3$ such that*

$$\left|\frac{g(t,k)}{f(t,k)} - 1\right| \leqslant \frac{d_3}{|k|^{\mu}} \tag{156}$$

*for all* $(\mathrm{Re}(t), k) \in (-\infty, 0] \times K(A)$.

Now, we are ready to demonstrate the existence of the asymptotic expansion (108) and (109) for impedance functions $p_+, p_-$ by converting Schrödinger equation into an integral equation (Lemma 4.5) and using the Neumann series (Lemma 4.6).

**Lemma 4.9.** *Suppose that impedance functions $p_+(x,k)p_-(x,k)$ are defined by (**31**) and (**32**).* Then,

$$p_+(x,k) = a_0(x) + \frac{a_1(x)}{ik} + \frac{a_2(x)}{(ik)^2} + \cdots + \frac{a_{\mu-1}(x)}{(ik)^{\mu-1}} + O(k^{-\mu}), \tag{157}$$

$$p_-(x,k) = b_0(x) + \frac{b_1(x)}{ik} + \frac{b_2(x)}{(ik)^2} + \cdots + \frac{b_{\mu-1}(x)}{(ik)^{\mu-1}} + O(k^{-\mu}) \tag{158}$$

*as $|k| \to \infty$. Here, $a = \{a_i : R \to C\}$, and $b = \{b_i : R \to C\}, i = 0, 1, 2, \cdots, \mu - 1$, with integer $\mu \geqslant 2$, are two sequences of complex functions.*

**Proof.** Combining (71) with (140), (141), (115), (116), (127), and (129), we obtain (158). Eq. (157) is derived similarly. □

Theorems 4.3 and 4.4 prove the statements (A) and (B) outlined in the beginning of Section 4.

**Theorem 4.3.** *Suppose that the functions $q, \rho, \gamma$ are in $c_0^2([0,1])$, and that $1 + q(x) - \alpha^2 > 0, \gamma(x) > 0, \rho(x) > 0$ for all $x \in R$, and $q'', \rho'', \gamma''$ are absolutely continuous. Suppose further that $D$ is the set of all pairs $(x,k)$, where $x \in R, k \in C^+$. Then*

*(a)* $\phi_+$ and $\phi_-$ are continuous functions of $(x,k)$ and analytic functions of k for all $x \in R$ and $k \in C$;
*(b)* $p_+$ and $p_-$ are continuous functions of $(x,k)$ and analytic functions of k in D;
*(c) For all $(x,k) \in D$,*

$$p_+(x,k) = \frac{1}{\rho(x)} \cdot \sqrt{1 + q(x) + i\gamma - \alpha^2} - \frac{1}{ik} \cdot \frac{\rho(x) \cdot (q'(x) + i\gamma'(x)) - 2 \cdot (1 + q(x) + i\gamma(x) - \alpha^2) \cdot \rho'(x)}{4 \cdot \rho^2(x) \cdot (1 + q(x) + i\gamma(x) - \alpha^2)} + O(k^{-2}), \tag{159}$$

$$p_-(x,k) = \frac{1}{\rho(x)} \cdot \sqrt{1 + q(x) + i\gamma - \alpha^2} + \frac{1}{ik} \cdot \frac{\rho(x) \cdot (q'(x) + i\gamma'(x)) - 2 \cdot (1 + q(x) + i\gamma(x) - \alpha^2) \cdot \rho'(x)}{4 \cdot \rho^2(x) \cdot (1 + q(x) + i\gamma(x) - \alpha^2)} + O(k^{-2}). \tag{160}$$

**Proof.** We only give the proof for $\phi_-, p_-$ since the proof for $\phi_+, p_+$ is very similar. We introduce two auxiliary functions $\hat{\phi}, \check{\phi}$: $R \times C^+ \to C$ by the formulae

$$\hat{\phi}(x,k) = \phi_-(x,k) - 1, \tag{161}$$

$$\check{\phi}(x,k) = \frac{\phi'_-(x,k)}{\rho(x)} + \frac{ik}{\rho_1} \cdot \sqrt{1 + q_1 + i\gamma_1 - \alpha^2}, \tag{162}$$

and combining (161) and (162) with (13) and initial conditions (33) and (34), we obtain the linear first order ODEs

$$\hat{\phi}'(x,k) = \rho(x)\check{\phi}(x,k) - ik\sqrt{1 + q_1 + i\gamma_1 - \alpha^2}\,\frac{\rho(x)}{\rho_1}, \tag{163}$$

$$\check{\phi}'(x,k) = -\frac{k^2}{\rho(x)}(1 + q(x) + i\gamma(x) - \alpha^2)(\hat{\phi}(x,k) + 1) \tag{164}$$

subject to the initial conditions

$$\hat{\phi}(0,k) = 0, \tag{165}$$

$$\check{\phi}(0,k) = 0. \tag{166}$$

According to Lemma 3.2, $\hat{\phi}, \check{\phi}$ are continuous functions of $x$ and analytic functions of $k$ for all $x \in R$ and $k \in C$, from which part (a) follows immediately. Similarly, we obtain part (b) by combining part (a) with (32) and the fact that $\phi_-(x,k) \neq 0$ for all $(x,k) \in D$ (see Lemma 4.3). The expansion (160) follows immediately from Lemmas 4.9 and 4.4. □

**Corollary 4.10.** *Under the conditions of the preceding theorem, there exist positive numbers $c_1, c_2$ such that*

$$\left| e^{ik\int_t^x p_+(\tau,k)\rho(\tau)d\tau} \right| \leqslant c_1, \tag{167}$$

$$\left| e^{ik\int_t^x p_-(\tau,k)\rho(\tau)d\tau} \right| \leqslant c_2 \tag{168}$$

*for all $t, x \in [0,1], k \in R$, or for all $0 \leqslant t \leqslant x \leqslant 1, k \in C^+$.*

**Proof.** Due to parts (b) and (c) of Theorem 4.3, the real part of the functions

$$\mathrm{Re}\left(ik\int_t^x p_+(\tau,k)d\tau\right) \leqslant c_3, \tag{169}$$

$$\mathrm{Re}\left(ik\int_t^x p_-(\tau,k)d\tau\right) \leqslant c_4, \tag{170}$$

where $c_3$ and $c_4$ does not depend on $t$ or $x$ for $t,x \in [0,1], k \in R$, or for all $0 \leqslant t \leqslant x \leqslant 1$, from which (167) and (168) follow immediately. □

**Theorem 4.4.** *Suppose that the functions $q,\gamma,\rho$ are in $c_0^m([0,1])$ with integer $m \geqslant 2, q^{(m)}, \rho^{(m)}, \gamma^{(m)}$ are absolutely continuous and $1+q(x)-\alpha^2 > 0, \gamma(x) > 0, \rho(x) > 0$ for all $x \in R$. Then there exists a positive real number a such that*

$$|p_+(x,-k) - p_-(x,k)| \leqslant \frac{a}{|k|^m} \tag{171}$$

*for all $(x,k) \in R \times C^+$.*

**Proof.** Due to (71) and (72),

$$p_+(x,-k) - p_-(x,k) = \sqrt{1+q(x)+i\gamma(x)-\alpha^2}\cdot\frac{1}{-ik\rho(x)}\cdot\left(\frac{\psi'_+(t,-k)}{\psi_+(t,-k)} - \frac{\psi'_-(t,k)}{\psi_-(t,k)}\right). \tag{172}$$

Combining Lemmas 4.7 and 4.8, and Eqs. (33) and (34) yields that (172) is true for all $x \notin (0,1)$. In order to prove the theorem for $x \in (0,1)$, we observe that $p_+(x,-k)$ and $p_-(x,k)$ obey the same Riccati equation (80) due to (79) and (80). Thus, the difference $s(x,k) = p_+(x,-k) - p_-(x,k)$ satisfies the ODE

$$s'(x,k) = ik\rho(x)(p_+(x,-k) + p_-(x,k))s(x,k). \tag{173}$$

Clearly, the solution to (173) is

$$s(x,k) = e^{ik\int_0^x (p_+(t,-k)+p_-(t,k))\rho(t)dt}s(0,k). \tag{174}$$

Due to (167) and (168), there exists constant $b > 0$ such that

$$\left|e^{ik\int_0^x (p_+(t,-k)+p_-(t,k))\rho(t)dt}\right| < b \tag{175}$$

for all $(x,k) \in [0,1] \times R$. Due to (82) and (71), and Lemma 4.8, there exists a positive number $c$ such that for all $k \in R$,

$$|s(0,k)| = |p_+(0,-k) - p_-(0,k)| = \left|p_+(0,-k) - \frac{\sqrt{1+q_1+i\gamma_1-\alpha^2}}{\rho_1}\right| \leqslant \frac{c}{|k|^m}. \tag{176}$$

Now, (171) follows immediately from (174) and (176). □

### *4.3. Trace formula*

In this section, we prove Theorem 4.5, which is the principal analytical tool of this paper. Theorem 4.5 describes what are known as the trace formulae for the impedance functions $p_+, p_-$ in the context of varying density, sound speed, and attenuation.

**Theorem 4.5.** Trace formula. Suppose that the functions $q,\rho,\gamma$ are in $c_0^m([0,1]), m \geqslant 2, q^{(m)}, \gamma^{(m)}, \rho^{(m)}$ are absolutely continuous and $1+q(x)-\alpha^2 > 0, \gamma(x) > 0, \rho(x) > 0$ for all $x \in R$. Then,

$$\rho(x)\cdot(q'(x)+i\gamma'(x)) - 2\cdot\rho'(x)\cdot(1+q(x)+i\gamma(x)-\alpha^2) = \frac{2}{\pi}(1+q(x)+i\gamma(x)-\alpha^2)\rho^2(x)\int_{-\infty}^{\infty}(p_+(x,k)-p_-(x,k))\,dk. \tag{177}$$

*Moreover, there exists a positive number c such that*

$$\begin{aligned}&|\rho(x)\cdot(q'(x)+i\gamma'(x)) - 2\cdot\rho'(x)\cdot(1+q(x)+i\gamma(x)-\alpha^2)\\ &\quad-\frac{2}{\pi}(1+q(x)+i\gamma(x)-\alpha^2)\rho^2(x)\int_{-a}^{a}(p_+(x,k)-p_-(x,k))dk| \leqslant \frac{c}{a^{(m-1)}}\end{aligned} \tag{178}$$

*for all $x \in R, a > 0$.*

**Proof.** Due to part (C) of Theorem 4.3, there exists $c > 0$ such that

$$\left|(p_+(x,k) - p_-(x,k)) - \left(-\frac{1}{ik}\cdot\frac{\rho(x)\cdot(q'(x)+i\gamma'(x)) - 2\cdot(1+q(x)+i\gamma(x)-\alpha^2)\cdot\rho'(x)}{2\cdot\rho^2(x)\cdot(1+q(x)+i\gamma(x)-\alpha^2)}\right)\right| \leqslant \frac{c}{|k|^2} \tag{179}$$

for all $(x,k) \in R \times C^+$. Denoting by $\Upsilon$ the upper half circle of radius $A$, with clockwise orientation, in the complex $k$-plane, i.e.,

$$\Upsilon = \{k | k \in C^+, |k| = A\}, \tag{180}$$

and noting that $p_+ - p_-$ is an analytical function of $k \in C^+$, we obtain

$$\int_{-A}^{A}(p_+(x,k) - p_-(x,k))dk = \int_{\Upsilon}(p_+(x,k) - p_-(x,k))dk. \tag{181}$$

Substituting (179) into (181), we have

$$\begin{aligned}&\frac{2}{\pi}(1+q(x)+i\gamma(x)-\alpha^2)\rho^2(x)\cdot\int_{-A}^{A}(p_+(x,k)-p_-(x,k))dk\\ &\quad = \rho(x)\cdot(q'(x)+i\gamma'(x)) - 2\cdot\rho'(x)\cdot(1+q(x)+i\gamma(x)-\alpha^2) + O(k^{-1})\end{aligned} \tag{182}$$

from which (177) follows immediately.

In order to prove (178), we rewrite (177) as

$$\begin{aligned}&\rho(x)\cdot(q'(x)+i\gamma'(x)) - 2\cdot\rho'(x)\cdot(1+q(x)+i\gamma(x)-\alpha^2)\\ &\quad = \frac{2}{\pi}(1+q(x)+i\gamma(x)-\alpha^2)\rho^2(x)\cdot\int_{-a}^{a}(p_+(x,k)-p_-(x,k))\,dk + I(a)\end{aligned} \tag{183}$$

with $I(a)$ given by the formula

$$\begin{aligned}I(a) &= \frac{2}{\pi}(1+q(x)+i\gamma(x)-\alpha^2)\rho^2(x)\cdot\left(\int_{-\infty}^{-a}+\int_{a}^{\infty}\right)(p_+(x,k)-p_-(x,k))\,dk\\ &= \frac{2}{\pi}(1+q(x)+i\gamma(x)-\alpha^2)\rho^2(x)\cdot\left(\int_{-\infty}^{-a}+\int_{a}^{\infty}\right)(p_+(x,-k)-p_-(x,k))\,dk,\end{aligned} \tag{184}$$

and using (171), we obtain a constant $c$ such that

$$|I(a)| \leqslant \frac{c^{(m-1)}}{a}, \tag{185}$$

from which (178) follows immediately. □

**Remark 4.6.** Identity (177) is directly used to reconstruct functions $\rho, q$, and $\gamma$ in our algorithm. In particular,

$$\rho'(x) = \frac{\mathrm{Re}(F(\alpha_1)) - \mathrm{Re}(F(\alpha_2))}{2(\alpha_1^2 - \alpha_2^2)}, \tag{186}$$

$$q'(x) = \frac{\mathrm{Re}(F(\alpha_1))(1+q(x)-\alpha_2^2) - \mathrm{Re}(F(\alpha_2))(1+q(x)-\alpha_1^2)}{\rho(x)(\alpha_1^2-\alpha_2^2)}, \tag{187}$$

$$\gamma'(x) = \frac{\mathrm{Im}(F(\alpha_1))\cdot(\alpha_1^2-\alpha_2^2) + \gamma(x)\cdot(\mathrm{Re}(F(\alpha_1)) - \mathrm{Re}(F(\alpha_2)))}{\rho(x)(\alpha_1^2-\alpha_2^2)}, \tag{188}$$

where

$$F(\alpha) = \frac{2}{\pi}(1+q(x)+i\gamma(x)-\alpha^2)\rho^2(x)\int_{-a}^{a}(p_+(x,k)-p_-(x,k))dk, \tag{189}$$

and $\alpha_1, \alpha_2$ represent two different incidence angles. Using more than two $\alpha$'s would lead to an overdetermined system of equations, and can be used to control the effects of noise.

**Remark 4.7.** Strictly speaking, most of the mathematical proofs in this section only apply to acoustic medium without attenuation, i.e., $\gamma(x) = 0$ for all $x \in R$, because of the violation of Eq. (124) for nonzero attenuation. However, numerical experiments in Section 6 indicate that our scheme still works for the case of small attenuation, i.e. $|\gamma(x)| \ll |1 + q(x) - \alpha^2|$.

## 5. The algorithm

This section describes the algorithm of the present paper, estimates its computational cost, and discusses the implementation in some detail.

*5.1. Description of the algorithm*

In this section, we describe a reconstruction algorithm for the scalar Helmholtz equation in layered acoustic media

$$\phi''_{\pm}(x,k) - \frac{\rho'(x)}{\rho(x)} \cdot \phi'_{\pm}(x,k) + k^2 \cdot (1 + q(x) + i \cdot \gamma(x) - \alpha^2) \cdot \phi_{\pm}(x,k) = 0 \tag{190}$$

subject to the initial conditions

$$\phi_{+}(x,k) = e^{ik\sqrt{1+q_2+i\gamma_2-\alpha^2}x} \quad \text{for all } x \geqslant 1, \tag{191}$$

$$\phi_{-}(x,k) = e^{-ik\sqrt{1+q_1+i\gamma_1-\alpha^2}x} \quad \text{for all } x \leqslant 0. \tag{192}$$

In (190)–(192), $x$ is a real number, $k$ is a complex number in the upper half plane, $\alpha$ is the sine of the angle of incidence with respect to the normal to the interface of layers, $\phi_+$ and $\phi_-$ are the scalar fields associated with right-going and left-going waves, respectively; the parameters to be recovered in this algorithm are the density $\rho$, potential $q$, and attenuation $\gamma$ of the layered media. We assume $\rho, q, \gamma \in c_0^m([0,1])$, i.e., $\rho, q, \gamma$ have $m$ continuous derivatives everywhere, and are defined by Eqs. (6)–(11).

As discussed in Sections 3 and 4, in order to reconstruct parameters $\rho, q, \gamma$, we consider a system of integro-differential equations

$$p'_{+}(x,k) = -ik\rho(x) \cdot \left(p_{+}^2(x,k) - \frac{1+q(x)+i\gamma(x)-\alpha^2}{\rho^2(x)}\right), \tag{193}$$

$$p'_{-}(x,k) = ik\rho(x) \cdot \left(p_{-}^2(x,k) - \frac{1+q(x)+i\gamma(x)-\alpha^2}{\rho^2(x)}\right), \tag{194}$$

$$\rho'(x) = \frac{\mathrm{Re}(F(\alpha_1)) - \mathrm{Re}(F(\alpha_2))}{2(\alpha_1^2 - \alpha_2^2)}, \tag{195}$$

$$q'(x) = \frac{\mathrm{Re}(F(\alpha_1))(1+q(x)-\alpha_2^2) - \mathrm{Re}(F(\alpha_2))(1+q(x)-\alpha_1^2)}{\rho(x)(\alpha_1^2-\alpha_2^2)}, \tag{196}$$

$$\gamma'(x) = \frac{\mathrm{Im}(F(\alpha_1)) \cdot (\alpha_1^2 - \alpha_2^2) + \gamma(x) \cdot (\mathrm{Re}(F(\alpha_1)) - \mathrm{Re}(F(\alpha_2)))}{\rho(x)(\alpha_1^2-\alpha_2^2)}, \tag{197}$$

with

$$F(\alpha) = \frac{2}{\pi}(1+q(x)+i\gamma(x)-\alpha^2)\rho^2(x)\int_{-a}^{a}(p_{+}(x,k) - p_{-}(x,k))dk \tag{198}$$

subject to the initial conditions

$$p_{+}(0,k) = p_0(k), \tag{199}$$

$$p_{-}(0,k) = \frac{\sqrt{1+q_1+i\gamma_1-\alpha^2}}{\rho_1}, \tag{200}$$

$$\rho(0) = \rho_1, \tag{201}$$

$$q(0) = q_1, \tag{202}$$

$$\gamma(0) = \gamma_1. \tag{203}$$

In (193) and (194), the impedance functions $p_+, p_- : (R, C^+) \to C$ are defined by the formulae

$$p_{+}(x,k) = \frac{\phi'_{+}(x,k)}{ik\rho(x)\phi_{+}(x,k)}, \tag{204}$$

$$p_{-}(x,k) = \frac{\phi'_{-}(x,k)}{-ik\rho(x)\phi_{-}(x,k)}. \tag{205}$$

Eqs. (193) and (194) are Riccati equations obtained directly from the Helmholtz equation (190) and the definitions of impedance functions (204) and (205); Eqs. (195)–(197) are known as trace formulae, connecting the Fourier components of the solutions of the Helmholtz equation to the parameters of the scattering objects to be recovered.

Our scheme amounts to solving numerically a self-contained set of ODEs, i.e., (193)–(197), subject to the initial conditions (199)–(203). In this paper, the ODE solver from [9] is used.

As we shall see in Section 6, for sufficiently large $a$, the system of ODEs (193)–(197) has a unique solution for all $x \in [0,1]$, and this solution is stable with respect to small perturbations of the initial data $p_0(k)$. The inversion algorithm is $(m-1)$th-order convergent for all three parameters $\rho, q, \gamma$ with $m$ the smoothness of $\rho, q$, and $\gamma$.

### 5.2. Implementation

In implementing the algorithm stated above, the integral

$$\int_{-a}^{a}(p_+(x,k)-p_-(x,k))dk \tag{206}$$

in (198) is approximated by the trapezoidal rule $T_n$, i.e.,

$$T_n(p_+(x,k)-p_-(x,k)) = h\sum_{j=-M+1}^{M-1}(p_+(x,k_j)-p_-(x,k_j)) + \frac{h}{2}((p_+(x,-a)-p_-(x,-a))+(p_+(x,a)-p_-(x,-))) \tag{207}$$

with $h=a/M, k_j=jh, j=-M,\dots,M$. Thus, the system of integro-differential Eqs. (193)–(197) subject to initial conditions (199)–(203) is converted into a system of $8M+7$ ODEs

$$p'_+(x,k_j) = -ik_j\rho(x)\cdot\left(p_+^2(x,k_j)-\frac{1+q(x)+i\gamma(x)-\alpha^2}{\rho^2(x)}\right), \tag{208}$$

$$p'_-(x,k_j) = ik_j\rho(x)\cdot\left(p_-^2(x,k_j)-\frac{1+q(x)+i\gamma(x)-\alpha^2}{\rho^2(x)}\right), \tag{209}$$

$$\rho'(x) = \frac{\mathrm{Re}(F(\alpha_1))-\mathrm{Re}(F(\alpha_2))}{2(\alpha_1^2-\alpha_2^2)}, \tag{210}$$

$$q'(x) = \frac{\mathrm{Re}(F(\alpha_1))(1+q(x)-\alpha_2^2)-\mathrm{Re}(F(\alpha_2))(1+q(x)-\alpha_1^2)}{\rho(x)(\alpha_1^2-\alpha_2^2)}, \tag{211}$$

$$\gamma'(x) = \frac{\mathrm{Im}(F(\alpha_1))\cdot(\alpha_1^2-\alpha_2^2)+\gamma(x)\cdot(\mathrm{Re}(F(\alpha_1))-\mathrm{Re}(F(\alpha_2)))}{\rho(x)(\alpha_1^2-\alpha_2^2)} \tag{212}$$

with

$$F(\alpha) = \frac{2}{\pi}(1+q(x)+i\gamma(x)-\alpha^2)\rho^2(x)\cdot T_{2M+1}(p_+(x,k)-p_-(x,k)), \tag{213}$$

and subject to the initial conditions

$$p_+(0,k_j) = p_0(k_j), \tag{214}$$

$$p_-(0,k_j) = \frac{\sqrt{1+q_1+i\gamma_1-\alpha^2}}{\rho_1}, \tag{215}$$

$$\rho(0) = \rho_1, \tag{216}$$

$$q(0) = q_1, \tag{217}$$

$$\gamma(0) = \gamma_1. \tag{218}$$

**Remark 5.1.** In the numerical examples in Section 6, the values of the initial impedance functions $p_0(k_j), j=-M,\dots,M$, required for the reconstruction scheme, are provided by solving forward scattering problems, namely, $4M+1$ independent ODEs

$$\phi''(x,k_j) - \frac{\rho'(x)}{\rho(x)}\cdot\phi'(x,k_j) + k_j^2\cdot(1+q(x)+i\cdot\gamma(x)-\alpha^2)\cdot\phi(x,k_j) = 0 \tag{219}$$

subject to the boundary conditions

$$\phi_+(x,k_j) = e^{ik_j\sqrt{1+q_2+i\gamma_2-\alpha^2}x} \quad \text{for all } x \geqslant 1 \tag{220}$$

for $k_j = j\cdot\frac{a}{M}, j=-M,\dots,M$ and $\alpha=\alpha_1,\alpha_2$. Again, we used the ODE solver in [9].

**Remark 5.2.** Due to Observation 3.3, for all $x,k\in R$,

$$\overline{p_+(x,k)} = p_+(x,-k), \tag{221}$$

$$\overline{p_-(x,k)} = p_-(x,-k), \tag{222}$$

thus, the integral

$$\int_{-a}^{a}(p_+(x,k)-p_-(x,k))dk \tag{223}$$

in (198) is equal to

$$2\cdot\int_0^a \mathrm{Re}(p_+(x,k)-p_-(x,k))dk. \quad (224)$$

Therefore, the dimensions of the system of ODEs we consider (see Eqs. (208)–(212) is reduced to $4M+7$ from $8M+7$.

### 5.3. Complexity analysis

The time cost of the inverse scheme is of the order $O(N_k\cdot N_z)$, where $N_k$ is the number of measurements in the frequency domain, and $N_z$ is the number of nodes in the space domain, since the computational cost for the ODE solver we use is proportional to the dimension of the ODE system ($N_k$ in our case) and the number of discretization points in the space domain ($N_z$).

Further, the storage requirements of the algorithm are also determined by $N_k$ and $N_z$, and is of the form

$$S=O(K\cdot N_k)+O(N_z), \quad (225)$$

where $K$ is a constant determined by the precision required by the ODE solver in [9]. For single precision, $K=22$; for double precision, $K=60$.

## 6. Numerical examples

The algorithm of Section 5 has been implemented in Fortran 77 in double precision. In this section, we illustrate the performance of the scheme as applied to several different classes of scattering objects, from Gaussian to discontinuous staircase-shaped ones. The experiments were carried out on a 2.8GHz Pentium D desktop with 2Gb of RAM and an L2 cache of 1 Mb. The calculations reported in Example 1 were carried out with a requested accuracy of $10^{-16}$ in the ODE solver; the calculations reported in Examples 2–4 were carried out with a requested accuracy of $10^{-7}$.

In Examples 1 and 2, the scatterers satisfy the smoothness conditions of Theorem 4.3. In Examples 3, 3.1 and 3.2, the scatterers violate the smoothness conditions mildly, as the scatterers are continuous but their derivatives are not continuous. In Example 4, the scatterers strongly violate the smoothness conditions, as those scatterers are discontinuous. The headings of the Tables are defined as follows:

$a$ is the highest frequency used in the algorithm;
$h_k$ is the step size in the discretization of the frequency interval;
$N_x$ is the number of discretization points in $[-1,1]$;
$E^2_\rho, E^2_q, E^2_\gamma$ are the relative $L^2$ errors of $\rho, q, \gamma$;
$E^\infty_\rho, E^\infty_q, E^\infty_\gamma$ are the relative maximum errors of $\rho, q, \gamma$;
$t_{CPU}$ is the CPU time required in seconds.

**Example 1.** In this example, we reconstruct scattering parameters $\rho, q$, and $\gamma$ of the Gaussian distribution given by the formulae

$$\rho(x)=1000+500\cdot e^{-40x^2}, \quad (226)$$

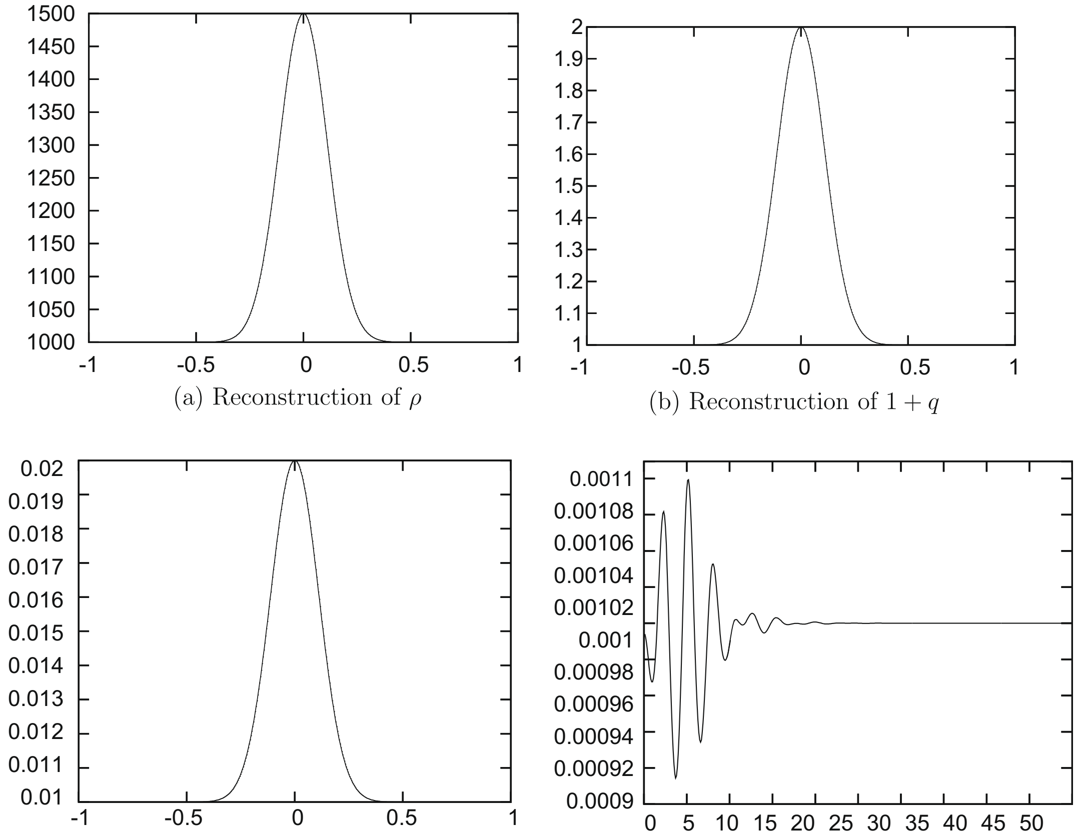
$$q(x)=e^{-40x^2}, \quad (227)$$

$$\gamma(x)=0.01+0.01\cdot e^{-40x^2}. \quad (228)$$

This is an example of scatterer whose $\rho, q$, and $\gamma$ are in $C_0^\infty$ in the interval $[-1,1]$. Table 1 illustrates the numerical behavior of the reconstruction algorithm, and Fig. 1 contains graphs of the exact and the recovered $\rho, q, \gamma\in C_0^\infty$ (they are indistinguishable on the graph) and the input impedance function $p_+(-1,k)$. In this example, the algorithm converges extremely rapidly as we expected.

**Table 1**
CPU time and accuracies for Example 1.

| $a$ | $h_k$ | $N_x$ | $E^2_\rho$ | $E^2_q$ | $E^2_\gamma$ | $t_{CPU}$ |
|---|---|---|---|---|---|---|
| 25 | 0.2 | 250 | 3.19E−05 | 6.95E−05 | 4.45E−05 | 3.5E+00 |
| 50 | 0.2 | 250 | 9.96E−06 | 2.05E−05 | 1.13E−05 | 7.0E+00 |
| 50 | 0.1 | 500 | 7.20E−09 | 1.48E−08 | 8.73E−09 | 1.9E+01 |
| 50 | 0.05 | 1000 | 7.13E−09 | 1.47E−08 | 8.60E−09 | 5.8E+01 |
| 100 | 0.2 | 500 | 9.54E−10 | 2.00E−09 | 1.28E−09 | 1.8E+01 |
| 100 | 0.1 | 1000 | 1.56E−12 | 3.22E−12 | 3.23E−12 | 5.9E+01 |
| 100 | 0.05 | 2000 | 4.21E−12 | 9.08E−12 | 9.08E−12 | 2.3E+02 |




(a) Reconstruction of $\rho$ (b) Reconstruction of $1+q$

(c) Reconstruction of $\gamma$ (d) Real Part of $p_+(-1,k)$

**Fig. 1.** Reconstruction of Example 1 with $a = 50$.

**Example 2.** In this example, we reconstruct a scatterer defined by the formulae

$$t = (x+1)\cdot\pi \tag{229}$$

$$\rho(x) = 1000 + 100\cdot\left((1-\cos(4t)) - \frac{22}{25}(1-\cos(5t)) + \frac{6}{49}(1-\cos(7t))\right), \tag{230}$$

$$q(x) = 0.4\cdot\left((1-\cos(3t)) - \frac{1215}{2783}(1-\cos(11t)) + \frac{7}{23}(1-\cos(12t))\right), \tag{231}$$

$$\gamma(x) = 0.01 + 0.003\cdot\left((1-\cos(2t)) - \frac{16}{21}(1-\cos(3t)) + \frac{5}{28}(1-\cos(4t))\right). \tag{232}$$

The scatterer is a $c_0^5$-function in $R$ with support in the interval $[-1,1]$. The performance of the algorithm is demonstrated in Table 2 and Fig. 2. As we can see from those tables, the convergence of the algorithm is actually better than the predicted fourth-order convergence.

**Example 3.** In this example, we construct a scatterer with discontinuous derivatives supported on $[-1,1]$, defined by the formulae

$$\rho(x) = 1000 + 500\cdot\sin(7x), \tag{233}$$

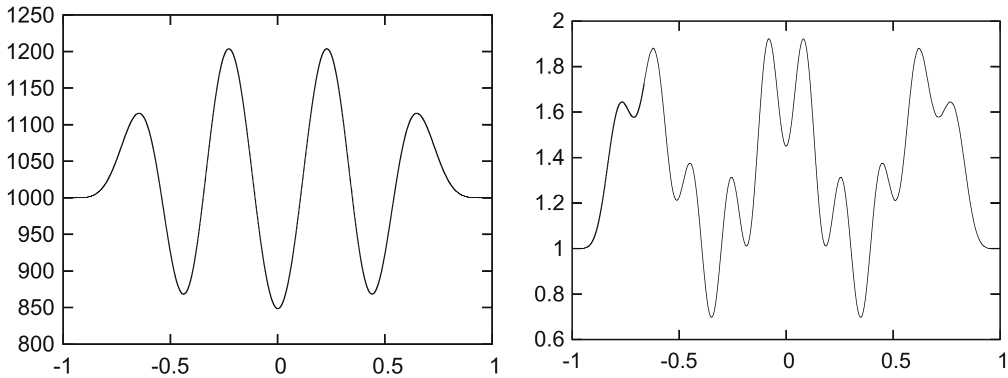
$$q(x) = 0.2\cdot\cos(30x)\cdot\frac{e^{3x}-e^{-3x}}{e^{3x}+e^{-3x}}, \tag{234}$$

$$\gamma(x) = 0.01 + 0.004\cdot\cos(20x)\cdot\frac{e^{11x}-e^{-11x}}{e^{11x}+e^{-11x}}. \tag{235}$$

Table 3 illustrates the numerical behavior of the algorithm; Fig. 3 demonstrates the exact and the reconstructed $\rho, q, \gamma$, and the input impedance function $p_+(-1,k)$. The algorithm is not convergent in this case, although the input impedance function $p_+(-1,k)$ decays for large frequencies. Further investigations (see [7]) show that, as we move with the ODE solver towards

**Table 2**
CPU time and accuracies for Example 2.

| $a$ | $h_k$ | $N_x$ | $E_\rho^2$ | $E_q^2$ | $E_\gamma^2$ | $t_{\text{CPU}}$ |
|---|---|---|---|---|---|---|
| 25 | 0.025 | 1000 | 7.95E−02 | 2.58E−01 | 6.78E−02 | 1.7E+01 |
| 50 | 0.025 | 1000 | 1.86E−02 | 6.57E−02 | 1.84E−02 | 4.1E+01 |
| 100 | 0.025 | 4000 | 1.87E−04 | 4.72E−04 | 3.34E−04 | 3.5E+02 |
| 200 | 0.0125 | 8000 | 8.50E−07 | 2.55E−06 | 9.45E−07 | 2.7E+03 |



(a) Reconstruction of $\rho$

the right boundary of the scattering structure, both the impedance function $p_+(x,k)$ and the integrand $p_+(x,k) - p_-(x,k)$ in the trace formulae (177) become extremely oscillatory and blow up as $k$ increases. This phenomenon is closely related to the interaction between non-smooth behavior of $\rho, q$ and the effect of attenuation. Examples 3.1 and 3.2 explores this phenomenon in more detail.

**Example 3.1.** This example uses the same $\rho$ and $q$ as in Example 3, but with zero attenuation; thus we have

$$\rho(x) = 1000 + 500 \cdot \sin(7x), \tag{236}$$

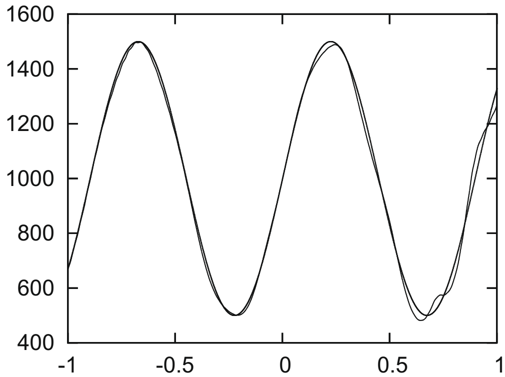
$$q(x) = 0.2 \cdot \cos(30x) \cdot \frac{e^{3x} - e^{-3x}}{e^{3x} + e^{-3x}}, \tag{237}$$

$$\gamma(x) = 0. \tag{238}$$

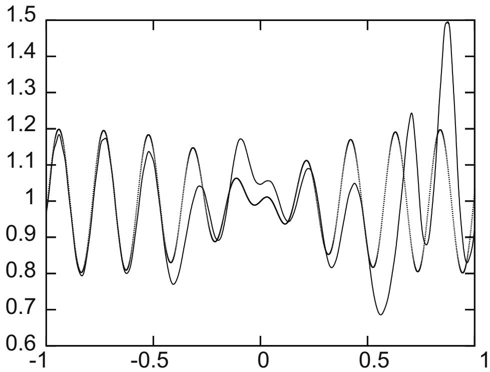
Table 4 illustrates the numerical behavior of the reconstruction algorithm. The algorithm exhibits linear convergence. Unlike Example 3, the integrand $p_+(x,k) - p_-(x,k)$ in the trace formulae (177), and thus in the ODE system, is not increasing with $k$ as the wave travels through the scattering object (see [7]).

**Table 3**
CPU time and accuracies for Example 3.

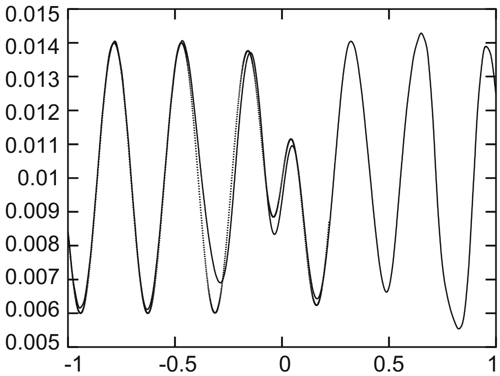
| $a$ | $h_k$ | $N_x$ | $E^2_\rho$ | $E^2_q$ | $E^2_\gamma$ | $t_{\mathrm{CPU}}$ |
|---|---|---|---|---|---|---|
| 25 | 0.025 | 1000 | 1.74E−02 | 4.10E−02 | 1.73E−01 | 1.7E+01 |
| 50 | 0.025 | 1000 | 3.22E−02 | 1.88E−01 | 1.66E−01 | 4.0E+01 |
| 100 | 0.025 | 2000 | 2.11E−02 | 1.30E−01 | 8.29E−02 | 1.7E+02 |
| 200 | 0.025 | 4000 | 1.71E−02 | 1.13E−01 | 1.62E−01 | 6.7E+02 |



(a) Reconstruction of $\rho$



(b) Reconstruction of $1+q$




**Example 3.2.** This example uses the same $q$ and $\gamma$ as in Example 3, but with a constant $\rho$; thus we have

$$\rho(x) = 1000, \tag{239}$$

$$q(x) = 0.2 \cdot \cos(30x) \cdot \frac{e^{3x} - e^{-3x}}{e^{3x} + e^{-3x}}, \tag{240}$$

$$\gamma(x) = 0.01 + 0.004 \cdot \cos(20x) \cdot \frac{e^{11x} - e^{-11x}}{e^{11x} + e^{-11x}}. \tag{241}$$

Table 5 illustrates the numerical behavior of the reconstruction algorithm. Linear convergence is observed for both $q$ and $\gamma$.

**Table 5**
CPU time and accuracies for Example 3.2.

| $a$ | $h_k$ | $N_x$ | $E_q^2$ | $E_\gamma^2$ | $E_q^\infty$ | $E_\gamma^\infty$ | $t_{CPU}$ |
|---|---|---|---|---|---|---|---|
| 25 | 0.05 | 1000 | 7.73E–02 | 1.35E–01 | 2.06E–01 | 4.67E–01 | 8.1E+00 |
| 50 | 0.1 | 1000 | 4.46E–02 | 1.20E–01 | 1.24E–01 | 4.63E–01 | 8.5E+00 |
| 100 | 0.1 | 2000 | 2.22E–02 | 7.67E–02 | 6.61E–02 | 3.41E–01 | 3.3E+01 |
| 200 | 0.1 | 4000 | 1.15E–02 | 1.74E–02 | 3.53E–02 | 6.12E–02 | 1.6E+02 |

**Table 6**
CPU time and accuracies for Example 4.

| $a$ | $h_k$ | $N_x$ | $E_\rho^2$ | $E_q^2$ | $E_\gamma^2$ | $t_{CPU}$ |
|---|---|---|---|---|---|---|
| 25 | 0.05 | 500 | 1.81E–02 | 3.77E–02 | 2.83E–02 | 4.2E+00 |
| 50 | 0.05 | 1000 | 2.55E–02 | 5.50E–02 | 2.25E–02 | 1.8E+01 |
| 100 | 0.1 | 1000 | 2.10E–02 | 4.53E–02 | 2.39E–02 | 1.8E+01 |
| 200 | 0.05 | 4000 | 3.77E–02 | 7.85E–02 | 5.06E–02 | 3.5E+02 |

**Example 4.** Here, we reconstruct a staircase-shaped scatterer defined by the formulae

$$\rho(x) = \begin{cases} 1050 & x \in (-\infty, -0.8] \\ 1150 & x \in (-0.8, -0.4] \\ 1250 & x \in (-0.4, 0.0] \\ 1350 & x \in (0.0, 0.4] \\ 1300 & x \in (0.4, 0.8] \\ 1200 & x \in (0.8, \infty) \end{cases} \tag{242}$$

$$q(x) = \begin{cases} 0 & x \in (-\infty, -0.8] \\ 0.1 & x \in (-0.8, -0.6] \\ 0.2 & x \in (-0.6, -0.2] \\ 0.3 & x \in (-0.2, 0.2] \\ 0.2 & x \in (0.2, 0.8] \\ 0 & x \in (0.8, \infty) \end{cases} \tag{243}$$

$$\gamma(x) = \begin{cases} 0.01 & x \in (-\infty, -0.8] \\ 0.012 & x \in (-0.8, -0.6] \\ 0.01 & x \in (-0.6, -0.2] \\ 0.008 & x \in (-0.2, 0.2] \\ 0.007 & x \in (0.2, 0.6] \\ 0.008 & x \in (0.6, 0.8] \\ 0.009 & x \in (0.8, \infty) \end{cases} \tag{244}$$

The numerical results are shown in Table 6, and Fig. 4. The algorithm does not converge in this situation.The following observations can be made from the tables above, and from other numerical experiments performed by us.

1. For scatterers satisfying the conditions of Theorem 4.3 (Examples 1 and 2), the numerical algorithm of Section 5 displays convergence of order $m-1$, where $m$ is the smoothness of the scatterer; the CPU time required is proportional to $N_k \cdot N_x$, where $N_k$ and $N_x$ are the numbers of discretization points in frequency and space domain, respectively.
2. For scatterers violating the conditions of Theorem 4.3 mildly, the algorithm does not converge. However, the algorithm exhibits linear convergence (see Examples 3.1 and 3.2) for the following two particular categories of scatterers violating the conditions of Theorem 4.3 mildly,
   A. $\rho, q$ are continuous but their derivatives are not, and $\gamma = 0$,
   B. $q, \gamma$ are continuous but their derivatives are not, and $\rho$ is a constant.
3. When the scatterer is discontinuous (Example 4), the algorithm produces results demonstrated in Fig. 4. The oscillatory behavior near the discontinuities is similar to the well-known Gibbs phenomenon. In general, the algorithm is not convergent for such scatterers. However, for scatterers of the following categories
   A. $q$ is discontinuous, $\rho$ is a constant, $\gamma = 0$,

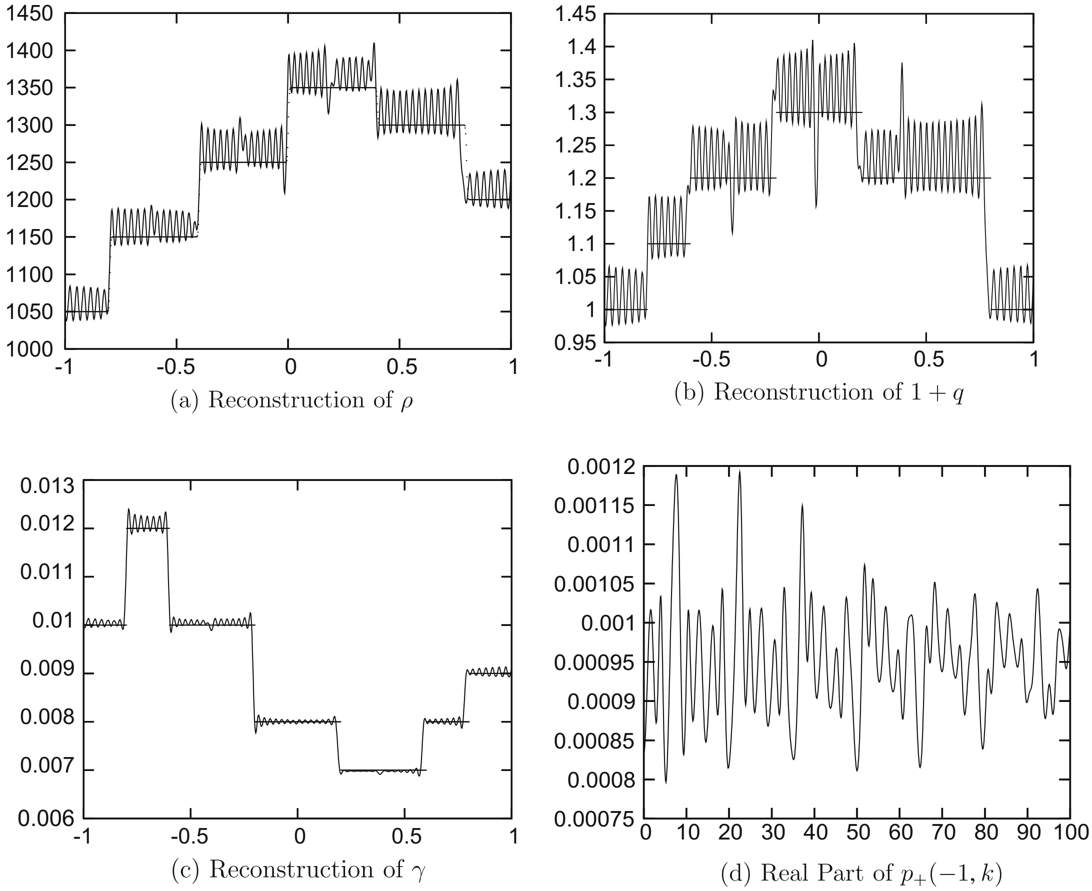



**Fig. 4.** Reconstruction of Example 4 with $a = 100$.

B. $\gamma$ is discontinuous, $\rho$ and $q$ are constants, the convergence of the algorithm is of the order $O\left(\frac{1}{\sqrt{a}}\right)$, where $a$ is the highest frequency.

4. When the initial data is perturbed (see [7] for detail), the error of the reconstruction is proportional to the magnitude of the perturbation, and the proportionality coefficient is 1.

## Acknowledgment

The authors would like to thank Mark Tygert for many useful discussions.

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